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# A Note on Topological M5-Branes and String–Fivebrane Duality

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**Abstract:** We derive the stability conditions for the M5-brane in topological M-theory using  $\kappa$ -symmetry. The non-linearly self-dual 3-form on the world-volume is necessarily non-vanishing, as is the case also for the 2-form field strengths on coisotropic branes in topological string theory. It is demonstrated that the self-duality is consistent with the stability conditions, which are solved locally in terms of a tensor in the representation  $\mathbf{6}$  of  $SU(3) \subset G_2$ . The double dimensional reduction of the M5-brane is the D4-brane, and its direct reduction is an NS5-brane. We show that the equation of motion for the 3-form on the NS5-brane wrapping a Calabi–Yau space is exactly the Kodaira–Spencer equation, providing support for a string–fivebrane duality in topological string theory.

## 1. INTRODUCTION AND CONCLUSIONS

The purpose of this paper is to formulate and examine the stability conditions (generalised calibration relations) for M5-branes in the topological M-theory formulated in ref. [1] (see also refs. [2,3]). The stability conditions, which have been discussed from a world sheet point of view for D-branes in string theory in ref. [4] and for topological string theory in refs. [5,6,7], can also be seen as a direct consequence of calibration [8] or demanding supersymmetry [9,10,11,12]. As is the case *e.g.* for the D4-brane in the A-model, the stability conditions demand non-vanishing world-volume field strength. Here we derive the corresponding stability conditions for the M5-brane in topological M-theory and its close relative the NS5-brane in the topological A-model. This is achieved using the  $\kappa$ -symmetric top-form formulation applied to the physical M5-brane in ref. [13]. In this approach there is in the 7-dimensional  $G_2$  superspace, apart from the super-4-form field strength, also a super-7-form field strength obeying the appropriate Bianchi identities, but without a bosonic component.

The M5-brane is, apart from the topological membrane constructed in ref. [14] (for a different approach see ref. [15,16]), the only brane present in topological M-theory<sup>†</sup>. Their direct and double dimensional reductions on a circle to a Calabi–Yau space give all NS-branes and D-branes in the A-model save for the isotropic D-branes with one-dimensional world sheets introduced in ref. [7] which should probably be viewed as Kaluza–Klein modes.

We proceed to demonstrate how the direct reduction of the M5-brane on  $CY \times S^1$  gives the NS5-brane in the topological A-model introduced in ref. [18,19] (see refs. [20,21] for a review of topological string theory), whose world-volume inherits the dynamical Kodaira–Spencer deformation theory [22] from the M5-brane. The related connection between the M5-brane instantons in the physical M-theory and Kodaira–Spencer theory was first pointed out in ref. [10]. The double reduction will give the D4-brane, with the stability conditions formulated by Kapustin and Li [6] (although that correspondence is not shown in the present paper). The NS5-brane provides a precise description of how duality between Kähler gravity [23] and Kodaira–Spencer theory [24], describing deformations of the Kähler and complex structures, respectively, is realised in the A-model as a “string–fivebrane duality” [25]. A forthcoming paper [26] will extend the discussion to the full sets of D-branes and RR fields in the A- and B-models.

Related conjectures have been made earlier. In ref. [27] Dijkgraaf, Verlinde and Vong used T-duality to relate the partition function on coinciding NS5-branes (with linear self-duality) in the A-model to a B-model calculation. S-duality, relating the A- and B-models on the same manifold, for topological strings, was conjectured on a twistorial CY by Neitzke and

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<sup>†</sup>  $G_2$  target spaces occur also in the topological string constructed in ref. [17]; its relation to topological M-theory is, however, unclear to us.

Vafa [20], and clarified, mainly using D-instantons, by Nekrasov, Ooguri and Vafa in ref. [28], where the existence of the topological NS5-brane was also pointed out. The relevance of the calculation of ref. [27] in this context was observed in ref. [29]. Gerasimov and Shatashvili, in their paper pointing towards a topological M-theory [2], relate Kodaira–Spencer theory to a 7-dimensional theory. Mariño et al. [10] derive conditions for  $D = 11$  M5-branes wrapping a Calabi–Yau space to preserve supersymmetry, and derive the Kodaira–Spencer equation. We comment to the relation of the present paper to the latter work in section 3.

## 2. TOPOLOGICAL M5-BRANES

The reduction of topological M-theory on a circle contains the A-model [1]. The presence in the A-model of a D4-brane and an NS5-brane implies that there has to be a 5-brane in topological M-theory. The purpose of this section is to derive, using superspace techniques and  $\kappa$ -symmetry, the stability conditions for this topological M5-brane, and to demonstrate the consistency between these conditions and the non-linear self-duality for the 3-form field strength on the brane. Open topological membranes have boundaries on the 5-brane, just as fundamental strings end on D-branes and D-branes on NS5-branes in the A-model.

We work in a superspace with 16 real fermionic directions and R-symmetry group  $SL(2)$ . This is half the number of fermionic coordinates compared to superstring theory or M-theory, and appropriate for the formulation of a topological 7-or 6-dimensional theory<sup>†</sup>. The dimension-0 components of the torsion and the 4-form field strength are

$$\begin{aligned} T_{\alpha I, \beta J}^a &= 2\varepsilon_{IJ}\gamma_{\alpha\beta}^a, \\ H_{ab, \alpha I, \beta J} &= 2\varepsilon_{IJ}(\gamma_{ab})_{\alpha\beta}. \end{aligned} \tag{2.1}$$

The real  $\gamma$ -matrices, which can be viewed as imaginary unit octonions multiplying octonionic spinors of  $Spin(7)$ , square to  $-1$ . For details on  $\gamma$ -matrices etc., we refer to the Appendix and to ref. [14].

Even though there is no bosonic 7-form field strength in the supergravity multiplet, there is a 7-form field strength on superspace, namely

$$\mathcal{H}_{abcde, \alpha I, \beta J} = 2\varepsilon_{IJ}(\gamma_{abcde})_{\alpha\beta}, \tag{2.2}$$

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<sup>†</sup> A more systematic formulation of topological theories embedded in supergravities with 16 supercharges will be given in a forthcoming paper [26].

with the Bianchi identity  $d\mathcal{H} + \frac{1}{2}H \wedge H = 0$ , following from the 7-dimensional Fierz identities. This presence of a superspace field strength that does not contain a purely bosonic part, or, more precisely, the absence of an invariant cohomologically non-trivial 6-form to calibrate the 6-cycle of the brane world-volume, is symptomatic for the cases of high-dimensional branes where non-vanishing world-volume field strength is demanded by the generalised calibration (stability) conditions.

We write an action for the 5-brane in complete analogy with ref. [13], the only difference being that the signature of the world-volume is euclidean,

$$S = \int d^6\xi \sqrt{g} \lambda [1 + \Phi(F) + (\star\mathcal{F})^2] , \quad (2.3)$$

where the field  $\lambda$  is a Lagrange multiplier and  $\Phi$  a functional to be determined.  $\mathcal{F}$  is the modified 6-form field strength of a 5-form potential  $\mathcal{A}$  and the 3-form  $F$  is the field strength of the 2-form  $A$ :

$$\begin{aligned} F &= dA - C , \\ \mathcal{F} &= d\mathcal{A} - \mathcal{C} - \frac{1}{2}A \wedge H \end{aligned} \quad (2.4)$$

where the pullbacked superfield potentials  $C$  and  $\mathcal{C}$  provide the coupling to the background. These field strengths are constructed with background gauge invariance as guideline. The Bianchi identities are  $dF = -H$ ,  $d\mathcal{F} = -\mathcal{H} + \frac{1}{2}F \wedge H$ . The action has of course to be supplemented by some self-duality condition. The advantage of actions of this type [30,31,32,13,33], with world-volume fields corresponding to all background fields the brane couples to, is (apart from complete control over background couplings and possible boundary conditions for lower-dimensional branes) that consistency of the non-linear self-duality relation is restrictive enough that demanding  $\kappa$ -symmetry gives its explicit form, which can be obtained without *a priori* specifying the function  $\Phi$ . At the same time, the corresponding projector on  $\kappa$  is derived, and  $\Phi$  can be constructed.

We define  $K^{ijk} \equiv \frac{\partial\Phi}{\partial F_{ijk}}$ . The equations of motion for  $A$ ,  $\mathcal{A}$  and  $\lambda$  are

$$\begin{aligned} d(\lambda\star K) - \lambda(\star\mathcal{F})H &= 0 , \\ d(\lambda\star\mathcal{F}) &= 0 , \\ 1 + \Phi + (\star\mathcal{F})^2 &= 0 , \end{aligned} \quad (2.5)$$

respectively. These must be consistent with the Bianchi identities, thus, combining the first two equations of motion with the Bianchi identity  $dF = -H$  we find  $K = (\star\mathcal{F}) \star F$ . By varying the action using  $\delta_\kappa F = -i_\kappa H$  and  $\delta_\kappa \mathcal{F} = -i_\kappa \mathcal{H} + \frac{1}{2}F \wedge i_\kappa H$  and inserting the relation between  $K$  and the field strengths, the projection matrix on  $\kappa$  and the non-linear

self-duality of the field strengths are obtained. We leave out the details, since they are in close parallel to ref. [13], and state the result. For the action to be invariant under the  $\kappa$ -symmetry the parameter  $\kappa$  must satisfy  $(1 - \Gamma)\kappa = 0$ , with

$$\Gamma = \frac{i}{N\sqrt{g}} \varepsilon^{ijklmn} \left[ \frac{1}{6!} \gamma_{ijklmn} + \frac{1}{2(3!)^2} F_{ijk} \gamma_{lmn} \right] \quad (2.6)$$

and  $N \equiv \sqrt{1 + \Phi}$ . The self-duality relation is

$$iN \star F_{ijk} = N^2 F_{ijk} + \frac{1}{2} q_{[i}^l F_{jk]l} , \quad (2.7)$$

where the sign choice  $\star \mathcal{F} = -i\sqrt{1 + \Phi} = -iN$  has been used. Here we have introduced the symmetric matrix  $k_{ij} = \frac{1}{2} F_i^{kl} F_{jkl}$  and the traceless  $q = k - \frac{1}{6} \text{tr } k$ . Inserting eq. (2.7), together with the Bianchi identities, into the equations of motion we find  $\Phi = -\frac{1}{6} \text{tr } k - \frac{1}{24} \text{tr } q^2 + \frac{1}{144} (\text{tr } k)^2$ . On the other hand, contracting the self-duality relation (2.7) with  $F^{ijk}$  gives  $\text{tr } q^2 = -24N^2(1 - N^2)$ , which by representation theory turns out to be the stronger relation  $q^2 = -4N^2(1 - N^2)\mathbb{1}$ . The equation of motion for the Lagrange multiplier  $\lambda$  now becomes

$$N^2 = 1 - \frac{1}{12} \text{tr } k . \quad (2.8)$$

This relation follows in fact also from  $\Gamma^2 = \mathbb{1}$ . Dualising the self-duality relation (and using all the known relations between  $N$ ,  $k$  and  $q$  as well as “ $\star(qF) = -q\star F$ ”) gives consistency.

After elimination of the top-form  $\mathcal{F}$ , we may write an action of a more standard type giving the same equations of motion,

$$S = \int d^6 \xi \sqrt{g(1 + \Phi)} + i \int (\mathcal{C} - \frac{1}{2} F \wedge C) . \quad (2.9)$$

Although this type of action (supplemented with some self-duality<sup>†</sup>) is less convenient as a starting point, the calibration relations we derive below has a clearer interpretation as relating kinetic and Wess–Zumino terms, as usual.

We are now ready to consider this M5-brane in a manifold with  $G_2$  holonomy, and look for 6-cycles that, together with the appropriate values of  $F$ , preserve supersymmetry. There is a covariantly constant spinor  $\eta^I$  (for each value of the  $SL(2)$  index  $I$ ), which we take to be the real part of the octonion. We expect the global supersymmetry to play the rôle of BRST charges, in analogy with the situation for the topological membrane of ref. [14].

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<sup>†</sup> Note that the implementation of the self-duality condition [13] can only be done on the level of the partition function, see [34, 35, 36].

Using the the methods of that reference it should be possible that the action (2.3) is not only BRST-invariant (supersymmetric) but also BRST-exact.

Using the explicit expressions for  $\gamma$ -matrices in terms of  $G_2$ -invariant tensors we have the action of  $\Gamma$  on the covariantly constant spinor:

$$\Gamma \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{i}{N\sqrt{g}} \varepsilon^{ijklmn} \left[ -\frac{1}{6!\sqrt{g}} \varepsilon_{ijklmn} \delta_7^\alpha + \frac{1}{2(3!)^2} F_{ijk} \sigma_{lmn} \right] , \quad (2.10)$$

where we for convenience have used a local basis where the direction  $dx^7$  is normal to the world-volume. The tensor  $\sigma$  is the covariantly constant  $G_2$ -invariant 3-form. The criterion for supersymmetry is that  $(1 - \Gamma)\eta = 0$ , which yields the stability conditions for the brane:

$$\begin{aligned} \frac{i}{2} F \wedge f^* \sigma &= N \text{Vol}_6 , \\ \frac{1}{2} F \wedge \star \sigma &= -\text{Vol}_7 , \\ F \wedge i_v f^* \star \sigma &= 0 , \end{aligned} \quad (2.11)$$

where  $\text{Vol}_6$  and  $\text{Vol}_7$  are the world-volume and space volume forms, respectively, and  $v$  is any world-volume vector. In order to solve these relations locally, and check their consistency, we parametrise the tensors using the local breaking of  $G_2$  to  $SU(3)$ , and use the standard relations  $\sigma = \text{Re } \Omega + \omega \wedge dx^7$ ,  $\star \sigma = -\text{Im } \Omega \wedge dx^7 - \frac{1}{2} \omega \wedge \omega$  (see appendix for conventions). At the moment this is not necessarily to be seen as the direct reduction to an A-model NS5-brane, although the local parametrisation suits this case. The  $SU(3)$ -covariant version of the stability conditions is

$$\begin{aligned} \frac{i}{2} F \wedge f^* \text{Re } \Omega &= N \text{Vol}_6 , \\ \frac{1}{2} F \wedge f^* \text{Im } \Omega &= \text{Vol}_6 , \\ F \wedge f^* \omega &= 0 . \end{aligned} \quad (2.12)$$

From the conditions (2.12) it follows immediately that  $F_{abc} = -\frac{1+N}{4} \Omega_{abc}$ ,  $F_{\bar{a}\bar{b}\bar{c}} = -\frac{1-N}{4} \bar{\Omega}_{\bar{a}\bar{b}\bar{c}}$  (we suppress explicit pullbacks from now on), and that  $g^{b\bar{c}} F_{ab\bar{c}} = 0$  and  $g^{a\bar{b}} F_{a\bar{b}\bar{c}} = 0$  (the last two equations leave only the representations  $\bar{\mathbf{6}}$  out of  $\bar{\mathbf{6}} \oplus \mathbf{3}$  in  $F_{(2,1)}$  and  $\mathbf{6}$  out of  $\mathbf{6} \oplus \bar{\mathbf{3}}$  in  $F_{(1,2)}$ ). It is not *a priori* clear that the stability conditions, derived from the  $G_2$  structure, are consistent with the self-duality relations. We will however show that this is indeed the case, and that, given the value of  $F_{(3,0)}$  from the stability condition, the self-duality relation dictates exactly the value of  $F_{(0,3)}$  given after eq. (2.12).

It is convenient to parametrise the non-linearly self-dual 3-form  $F$  in terms of a linearly self-dual one,  $h$ . It is straightforward to show that  $h_{ijk} = F_{ijk} + \frac{1}{2N(1+N)} q_i{}^l F_{jkl}$  satisfies  $i\star h = h$ . Forming the matrix  $r_{ij} = \frac{1}{2} h_i{}^{kl} h_{jkl}$ , the relations above give  $r = \frac{2}{N(1+N)} q$ , so the

relation between  $h$  and  $F$  becomes  $h_{ijk} = m_i^l F_{jkl}$ , where  $m = \mathbb{1} + \frac{1}{4}r$ . Inverting the matrix  $m$ ,  $m^{-1} = \frac{(1+N)}{2}(\mathbb{1} - \frac{1}{4}r)$ , finally gives the explicit parametrisation of  $F$  in terms of  $h$ ,

$$F_{ijk} = \frac{1+N}{2} \left( h_{ijk} - \frac{1}{4} r_i^l h_{jkl} \right) , \quad (2.13)$$

where the scalar  $N$  now is defined by  $r^2 = -16 \frac{1-N}{1+N} \mathbb{1}$ .

The general Ansatz for  $h$  in terms of  $SU(3)$  tensors contains a singlet in  $h_{(3,0)}$  ( $\xi$ ), a triplet  $\mathbf{3}$  in  $h_{(2,1)}$  and the representation  $\mathbf{6}$  in  $h_{(1,2)}$  ( $u$ ). It is clear that the triplet generates triplets in  $F$  violating the last equation in (2.12), so we set it to zero. The Ansatz becomes

$$\begin{aligned} h_{abc} &= \frac{1}{2} \xi \Omega_{abc} , \\ h_{ab\bar{c}} &= 0 , \\ h_{a\bar{b}\bar{c}} &= \frac{1}{2} u_a^{\bar{d}} \bar{\Omega}_{\bar{b}\bar{c}\bar{d}} , \\ h_{\bar{a}\bar{b}\bar{c}} &= 0 . \end{aligned} \quad (2.14)$$

The matrix  $r$  has the non-vanishing components  $r_{ab} = 4\xi u_a^{\bar{c}} g_{b\bar{c}}$ ,  $r_{\bar{a}\bar{b}} = \frac{1}{4} \bar{\Omega}_{\bar{a}}^{cd} \bar{\Omega}_{\bar{b}\bar{c}\bar{d}} u_c^{\bar{e}} u_d^{\bar{f}}$ . Calculating  $F$  from this Ansatz gives immediately  $F_{abc} = \frac{1+N}{4} \xi \Omega_{abc}$ , so  $\xi = 1$  by the stability conditions. We have  $\text{tr } r^2 = 96 \det u$  (note that  $\det u = \frac{1}{8 \cdot 3!} \bar{\Omega}^{abc} \bar{\Omega}_{\bar{a}\bar{b}\bar{c}} u_a^{\bar{a}} u_b^{\bar{b}} u_c^{\bar{c}}$ ), and thus  $\det u = \frac{1-N}{1+N}$ . The complete non-linearly self-dual tensor is

$$\begin{aligned} F_{abc} &= -\frac{1+N}{4} \Omega_{abc} , \\ F_{ab\bar{c}} &= \frac{1+N}{4} \bar{\Omega}_{\bar{c}\bar{d}\bar{e}} u_a^{\bar{d}} u_b^{\bar{e}} = \frac{1-N}{4} \Omega_{abd} (u^{-1})_{\bar{c}}^{\bar{d}} , \\ F_{a\bar{b}\bar{c}} &= \frac{1+N}{4} u_a^{\bar{d}} \bar{\Omega}_{\bar{b}\bar{c}\bar{d}} , \\ F_{\bar{a}\bar{b}\bar{c}} &= -\frac{1+N}{4} \bar{\Omega}_{\bar{a}\bar{b}\bar{c}} \det u = -\frac{1-N}{4} \bar{\Omega}_{\bar{a}\bar{b}\bar{c}} . \end{aligned} \quad (2.15)$$

We notice that the value of  $F_{(0,3)}$  consistent with the stability conditions is exactly the one that follows from non-linear self-duality. This concludes the check of algebraic consistency of the stability conditions (2.11) with the self-duality relation (2.7), and provides an explicit parametrisation for the following section.

### 3. NS5-BRANES IN THE A-MODEL AND KODAIRA–SPENCER THEORY

So far, the analysis is completely local and algebraic. We will show that the equation of motion (or equivalently, the Bianchi identity) for the 3-form is the Kodaira–Spencer equation. We will now suppose that the M5-brane actually winds a Calabi–Yau space, so that it

becomes an NS5-brane in the A-model. The components of  $dF = 0$  are (we assume that the RR field strengths vanish)

$$\begin{aligned} (dF)_{(1,3)} : \quad & \partial_a N - \bar{\partial}_b [(1+N)u_a^{\bar{b}}] = 0 , \\ (dF)_{(2,2)} : \quad & \bar{\Omega}^{acd} \partial_c [(1+N)u_d^{\bar{b}}] + \Omega^{\bar{b}\bar{c}\bar{d}} \bar{\partial}_{\bar{c}} [(1-N)(u^{-1})_{\bar{d}}^a] = 0 , \\ (dF)_{(3,1)} : \quad & \bar{\partial}_{\bar{a}} N + \partial_b [(1-N)(u^{-1})_{\bar{a}}^b] = 0 . \end{aligned} \quad (3.1)$$

It is straightforward to show, using  $dN = -\frac{1}{2}(1-N^2)\text{tr}(u^{-1}du)$ , that the first two equations imply the third. The first equation can be seen as a gauge-fixing condition, while the second one reads

$$\begin{aligned} 0 = \partial_{[a}[(1+N)u_{b]}^{\bar{c}}] + \bar{\partial}_{\bar{d}}[(1+N)u_{[a}^{\bar{d}}u_{b]}^{\bar{c}}] \\ = (\partial_{[a}N - \bar{\partial}_{\bar{d}}[(1+N)u_{[a}^{\bar{d}}]u_{b]}^{\bar{c}} + (1+N)(\partial_{[a}u_{b]}^{\bar{c}} - u_{[a}^{\bar{d}}\bar{\partial}_{\bar{d}}u_{b]}^{\bar{c}}) , \end{aligned} \quad (3.2)$$

which, using the gauge-fixing condition, implies that  $u$  fulfills the Kodaira–Spencer equation

$$\partial_{[a}u_{b]}^{\bar{c}} - u_{[a}^{\bar{d}}\bar{\partial}_{\bar{d}}u_{b]}^{\bar{c}} = 0 , \quad (3.3)$$

corresponding to the deformation of the complex structure encoded in the differential  $\partial' = dz^a(\partial_a - u_a^{\bar{b}}\bar{\partial}_{\bar{b}})$ .

The non-linearly self-dual closed 3-form  $F$  is exactly the deformation of the form  $\frac{1}{2}\Omega$  defining the complex structure. It will be linearly self-dual under the deformed metric. It is possible to be quite explicit about the deformed metric  $G$ , such that  $i\star_G F = F$ . From the form of the non-linear self-duality relation, it is clear that the metric  $G$  satisfies (using that the antisymmetry of  $G_{[i}^{\phantom{i}l}F_{jk]l}$  is automatic provided  $G$  is expressible in terms of  $F$ )

$$\frac{G^3}{\sqrt{\det G}} = N\mathbb{1} - \frac{1}{2N}q , \quad (3.4)$$

where contractions are made with the undeformed metric (which we for calculations have taken to be locally  $\mathbb{1}$ ). The right hand side has unit determinant. The expressions become more transparent if we use the normalised matrix  $s = \frac{1}{2N\sqrt{1-N^2}}q$  with  $s^2 = -\mathbb{1}$ . We then have  $(\det G)^{-1/2}G^3 = N\mathbb{1} - \sqrt{1-N^2}s = e^{-s\theta}$ , where  $\theta$  is defined by  $\cos\theta = N$ . The deformed metric is thus defined, up to a scale factor, by

$$(\det G)^{-1/6}G = e^{-\frac{1}{3}s\theta} . \quad (3.5)$$



It will of course be hermitean only with respect to the deformed complex structure.

We would like to comment on the relation to the treatment of the 11-dimensional M5-brane instantons winding on CY spaces of ref. [10]. The projection matrix on the  $\kappa$  parameter stated there does not contain the actual  $\Gamma$  of eq. (2.6), but only its linearisation in  $h$ , which is the projection arising from a superembedding treatment [37]. It was shown in ref. [38] how the two apparently different projections “ $\frac{1}{2}(1 - \Gamma)$ ” are related, and that they both project on the fermionic gauge degrees of freedom. Here we start from a topological M5-brane, in a superspace with 7 bosonic coordinates and half the number of fermions compared to M-theory, whose presence in topological M-theory is necessitated by the existence of D4- and NS5-branes in the A-model.

## APPENDIX: CONVENTIONS

In 7 euclidean dimensions, we use  $\gamma$  matrices that satisfy  $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$ , where the minus sign is necessary for real  $\gamma$ -matrices. The spinors are real  $\psi_\alpha^I$ , where  $\alpha = 1, \dots, 8$  and the  $I = 1, 2$  is an  $SL(2, R)$   $R$ -symmetry index [14].

For the 3-form  $\sigma$ , we use  $\sigma_{124} = 1$  and cyclic. On the CY space, with 3 complex dimensions, we use locally  $\Omega_{abc} = \varepsilon_{abc}$ , so that  $\Omega \wedge \bar{\Omega} = 8i\text{Vol}_6$ . We have  $g_{a\bar{b}} = \frac{1}{2}\delta_{a\bar{b}}$  and  $\omega_{a\bar{b}} = \frac{i}{4}\delta_{a\bar{b}}$ , so that  $\omega \wedge \omega \wedge \omega = -6\text{Vol}_6$ . The relations between 7-dimensional and 6-dimensional forms are

$$\begin{aligned}\sigma &= \text{Re } \Omega + \omega \wedge dx^7, \\ \star\sigma &= -\text{Im } \Omega \wedge dx^7 - \frac{1}{2}\omega \wedge \omega.\end{aligned}\tag{A.1}$$

The real 7-dimensional  $\gamma$  matrices encoded in the left multiplication of a spinor  $\lambda = \lambda^{\hat{\alpha}}e_{\hat{\alpha}}$  by an imaginary unit  $e_a$  are

$$\begin{aligned}(\gamma^a)_{\alpha\beta} &= \sigma^a_{\alpha\beta}, \\ (\gamma^a)_{0\alpha} &= \delta^a_\alpha.\end{aligned}\tag{A.2}$$

The Clifford algebra is spanned by the  $so(7)$ -invariant tensors  $\delta^{\hat{\alpha}}_{\hat{\beta}}$ ,  $(\gamma^a)^{\hat{\alpha}}_{\hat{\beta}}$ ,  $(\gamma^{ab})^{\hat{\alpha}}_{\hat{\beta}}$  and  $(\gamma^{abc})^{\hat{\alpha}}_{\hat{\beta}}$ , of which the first and last are symmetric and the second and third antisymmetric

matrices. The decomposition in terms of  $G_2$ -invariant tensors is

$$\begin{aligned}
\delta^{\hat{\alpha}}_{\hat{\beta}} &= \begin{bmatrix} 1 & 0 \\ 0 & \delta^{\alpha}_{\beta} \end{bmatrix}, \\
(\gamma^a)^{\hat{\alpha}}_{\hat{\beta}} &= \begin{bmatrix} 0 & \delta^a_{\beta} \\ -\delta^{a\alpha} & \sigma^{a\alpha}_{\beta} \end{bmatrix}, \\
(\gamma^{ab})^{\hat{\alpha}}_{\hat{\beta}} &= \begin{bmatrix} 0 & -\sigma^{ab}_{\beta} \\ \sigma^{ab\alpha} & -\star \sigma^{ab\alpha}_{\beta} - 2\delta^{ab}_{\alpha\beta} \end{bmatrix}, \\
(\gamma^{abc})^{\hat{\alpha}}_{\hat{\beta}} &= \begin{bmatrix} \sigma^{abc} & -\star \sigma^{abc}_{\beta} \\ -\star \sigma^{abc\alpha} & 6\delta^{[a}_{(\alpha} \sigma^{bc]}_{\beta)} - \delta^{\alpha}_{\beta} \sigma^{abc} \end{bmatrix}.
\end{aligned} \tag{A.3}$$

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